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# Individual eigenvalue distributions of chiral random two-matrix theory and the determination of $F_\pi$

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Abstract: Dirac operator eigenvalues split into two when subjected to two different external vector sources. In a specific finite-volume scaling regime of gauge theories with fermions, this problem can be mapped to a chiral Random Two-Matrix Theory. We derive analytical expressions to leading order in the associated finite-volume expansion, showing how individual Dirac eigenvalue distributions and their correlations equivalently can be computed directly from the effective chiral Lagrangian in the epsilon-regime. Because of its equivalence to chiral Random Two-Matrix Theory, we use the latter for all explicit computations. On the mathematical side, we define and determine gap probabilities and individual eigenvalue distributions in that theory at finite $N$, and also derive the relevant scaling limit as $N$ is taken to infinity. In particular, the gap probability for one Dirac eigenvalue is given in terms of a new kernel that depends on the external vector source. This expression may give a new and simple way of determining the pion decay constant $F_\pi$ from lattice gauge theory simulations.

Keywords: Matrix Models, Lattice QCD, Chiral Lagrangians.

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## Contents



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## 1. Introduction

One of the most challenging — and perhaps most interesting — problems associated with lattice gauge theory simulations of QCD is that of the chiral limit. Based on a variety of different approaches it is now possible to perform numerical simulations of the theory with two light dynamical quark flavours, at least in modest space-time volumes. By "light" quarks we mean quarks that are very close to the actual physical masses of the $u$ and $d$ quarks in QCD. Even if the masses of the physical $u$ and $d$ quarks had turned out to be much heavier (on the typical QCD scale $\Lambda_{\rm QCD}$), one would like to explore the chiral limit of the theory in it own right. This is because the theory in this limit separates into to two disjoint regimes, of which the low-energy part can be treated in a systematic manner by means of effective field theory: the chiral Lagrangian based on the spontaneous breaking of chiral symmetry. This low-energy theory of QCD with very light quarks can be understood in much the same way that the low-energy limit of QCD without quarks matches on to an effective string theory description, and both limits are of interest.

The so-called $\epsilon$-regime of QCD [1] is particularly useful for studying the chiral limit of QCD in finite volume. It is well known how a universality class of chiral Random Matrix Theory [2] provides an intriguing alternative description of the leading-order expressions for Dirac operator eigenvalue correlation functions in this regime, results that also can be derived directly from the low-energy effective field theory [3, 4]. Even the distributions

of individual Dirac operator eigenvalues follow from a systematic expansion in the chiral Lagrangian framework [5]. All of these analytical results depend on just one single low-energy constant of QCD, that of the infinite-volume chiral condensate $\Sigma$. From a lattice gauge theory viewpoint, this provides a new and unusual way of determining this low-energy constant of QCD by measuring the lowest-lying Dirac operator eigenvalues. For some numerical analyses see, *e.g.*, refs. [6, 7].

Recently, a new scheme was proposed which uses Dirac operator eigenvalues for determining the pion decay constant $F_\pi$ in a somewhat similar manner [8]. Based on the chiral Lagrangian formulation, the suggested method made use of a spectral 2-point function associated with two different Dirac operators,

$$\begin{aligned} D_1\psi_1^{(n)} &\equiv [\not{D}(A) + i\mu_1\gamma_0]\psi_1^{(n)} &&= i\lambda_1^{(n)}\psi_1^{(n)}, \\ D_2\psi_2^{(n)} &\equiv [\not{D}(A) + i\mu_2\gamma_0]\psi_2^{(n)} &&= i\lambda_2^{(n)}\psi_2^{(n)}, \end{aligned} \tag{1.1}$$

corresponding, in the case $\mu \equiv \mu_1 = -\mu_2$, to *imaginary isospin chemical potential*. Equivalently, the two Dirac operators (1.1) are simply in a constant background Abelian gauge field, but with different "charges". In the $\epsilon$-regime, the chemical potential $\mu$ couples directly to $F_\pi$ in the form of the finite-volume scaling variable $\hat{\mu} = \mu F_\pi\sqrt{V}$. Because the sensitivity to $\mu$ is quite drastic for the spectral 2-point function, this provides a clean method for extracting $F_\pi$. There is sensitivity to the parameter $\hat{\mu}$ (and hence $F_\pi$) also in other observables in the $\epsilon$-regime [9, 10]. Alternatively, one may use a real chemical potential to determine $F_\pi$ [11], with the same finite-volume scaling. The Dirac spectrum is complex in that case.

The chiral Lagrangian approach of ref. [8] can, to leading order in the $\epsilon$-regime, also be re-cast in terms of Random Matrix Theory, this time a Random Two-Matrix Theory [12]. All eigenvalue density correlations are equivalent in the two theories [8, 4]. One loop corrections to both $\Sigma$ [1] and $F_\pi$ [13] have been computed in the $\epsilon$-expansion. To that order they simply amount to finite-volume corrections to the infinite-volume quantities $\Sigma$ and $F_\pi$; the effective theory otherwise remains unchanged. It is of course important to know the size of these finite-size corrections if one wishes to determine $\Sigma$ and $F_\pi$ from the eigenvalues of the Dirac operator by means of lattice gauge theory simulations at finite volume. An alternative method for extracting $\Sigma$ and $F_\pi$ in the $\epsilon$-regime of QCD can be based on fits to vector and axial vector two-point correlations functions; also here finite-volume corrections are known analytically at sectors of fixed gauge field topology [14]. See the recent review [15] for a summary of these different approaches.

In the Random Matrix Theory formulation analytical computations are substantially simplified, and in ref. [12] all possible spectral density correlation functions associated with the two Dirac operators were found analytically. This includes all spectral functions in both the quenched and unquenched theory, and even all spectral correlation functions associated with "partially quenched" spectral correlation functions, where there is no back-reaction of the chemical potential on the gauge field configurations. This latter set of spectral correlation functions give the most fruitful way of extracting $F_\pi$ from lattice data since one can make use of ordinary configurations without chemical potential. Once all spectral correlation functions are known, one should in principle have all spectral data, and thus be

able to reconstruct *individual eigenvalue distributions* as well. Indeed, this was precisely what was found in ref. [5] for the case without imaginary chemical potential. In the first part of this paper we show that this is also the case here.[1] It was also shown in [18] that in the quenched theory in the limit of large chemical potential all spectral and individual eigenvalue correlations factorise into quenched one-matrix theory quantities.

In the second part of this paper we aim at determining in as closed form as possible the precise analytical expressions for individual eigenvalues distributions. Such expressions may turn out to be very useful alternatives for extracting $F_\pi$ from lattice simulations in those cases where only a very small number of "good" eigenvalues are available, and where it therefore may be difficult to construct the spectral 2-point correlation function with good statistics. Here we concentrate on the smallest eigenvalue, but provide the general framework for computing others. Based on the Random Two-Matrix Theory representation, we derive an explicit and quite compact representation for any finite $N$. Taking the scaling limit with $N \to \infty$, this provides the sought-for analytical expression for the lowest Dirac operator eigenvalue distribution in the appropriate finite-volume scaling regime. Remarkably, the final expression is not much more involved than the one without external vector sources.

From the point of view of mathematical physics, the resulting solution for the distribution of the smallest eigenvalue in the chiral Random Two-Matrix Theory is of interest in its own right. The solution cannot be mapped on to an analogous one-matrix theory, so the distribution is new and presumably corresponds to a new universality class that is parametrised by one real number $\hat{\delta}$. For this reason we include a rather detailed derivation, even though the resulting formula is all that is needed for the purpose of applications to lattice gauge theory simulations.

## 2. Eigenvalue correlations in chiral random two-matrix theory

We start by giving the theory we will solve for individual eigenvalue correlations, chiral perturbation theory in the epsilon regime with imaginary chemical potential

$$\mathcal{Z}_\nu^{(N_f)} = \int_{\mathrm{U}(N_f)} dU\, (\det U)^\nu e^{\frac{1}{4}VF_\pi^2 \mathrm{Tr}[U,B][U^\dagger,B]+\frac{1}{2}\Sigma V \mathrm{Tr}(\mathcal{M}^\dagger U+\mathcal{M}U^\dagger)}\,. \tag{2.1}$$

Here the matrix $B = \mathrm{diag}(\mu_1 \mathbf{1}_{N_1}, \mu_2 \mathbf{1}_{N_2})$ contains the two different chemical potentials, and $\mathcal{M} = \mathrm{diag}(m_1, \ldots, m_{N_f})$ is the quark mass matrix of the $N_1 + N_2 = N_f$ flavours.

This theory and all its spectral density correlation functions are completely equivalent to the chiral Two-Matrix Theory with imaginary chemical potential that was introduced in ref. [12]. The equivalence for the two-point function follows from [8], for all higher density correlations it was proven in [4]. It is defined as

$$\mathcal{Z}_\nu^{(N_f)} \sim \int d\Phi d\Psi\; e^{-N\mathrm{Tr}\left(\Phi^\dagger\Phi+\Psi^\dagger\Psi\right)} \prod_{f1=1}^{N_1} \det[\mathcal{D}_1 + m_{f1}] \prod_{f2=1}^{N_2} \det[\mathcal{D}_2 + m_{f2}]\,, \tag{2.2}$$

[1] A preliminary account of this was presented at a conference last year [16]. A comparison with Monte Carlo data from lattice gauge theory was presented at the same meeting [17].

where $\mathcal{D}_f$ is given by

$$\mathcal{D}_f = \begin{pmatrix} 0 & i\Phi + i\mu_f\Psi \\ i\Phi^\dagger + i\mu_f\Psi^\dagger & 0 \end{pmatrix} \quad , \quad f = 1, 2 \quad . \tag{2.3}$$

The operator remains anti-Hermitian because the chemical potentials are imaginary. Both $\Phi$ and $\Psi$ are complex rectangular matrices of size $N \times (N + \nu)$, where both $N$ and $\nu$ are integers. The index $\nu$ corresponds to gauge field topology in the usual way.

Referring to ref. [12] for details, we immediately write down the corresponding eigenvalue representation:

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)} = \int_0^\infty \prod_{i=1}^{N} \left( dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ \times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det\left[ I_\nu(2dN x_i y_j) \right] e^{-N \sum_i^N c_1 x_i^2 + c_2 y_i^2}, \end{aligned} \tag{2.4}$$

up to an irrelevant normalisation factor. Here the $x_i$'s and $y_i$'s are the real and non-negative entries in the diagonal matrices $X$ and $Y$, defined by

$$\begin{aligned} \Phi_1 \equiv \Phi + \mu_1 \Psi &= U_1 X V_1^\dagger \,, \\ \Phi_2 \equiv \Phi + \mu_2 \Psi &= U_2 Y V_2^\dagger \,. \end{aligned} \tag{2.5}$$

Because of this redefinition the matrices $\Phi_i$ become coupled in the exponent, leading to the above structure after integration of the unitary matrices $U_i$ and $V_i$. This leads to the following combinations of the two chemical potentials in eq. (2.4):

$$\begin{aligned} c_1 &= (1 + \mu_2^2)/\delta^2, & c_2 &= (1 + \mu_1^2)/\delta^2 \,, \\ d &= (1 + \mu_1\mu_2)/\delta^2, & 1 - \tau &= d^2/(c_1 c_2) \,, \\ \delta &= \mu_2 - \mu_1 \,, \end{aligned} \tag{2.6}$$

where the latter will become useful in section 4.

The joint probability distribution function which is proportional to the integrand in eq. (2.4) is defined as

$$\begin{aligned} \mathcal{P}_\nu^{(N_f)}(\{x\},\{y\};\{m_1\},\{m_2\}) = \frac{1}{\mathcal{Z}_\nu^{(N_f)}} \prod_{i=1}^{N} \left( (x_i y_i)^{\nu+1} \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ \times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det\left[ I_\nu(2dN x_i y_j) \right] e^{-N \sum_i^N c_1 x_i^2 + c_2 y_i^2}, \end{aligned} \tag{2.7}$$

where $\Delta_N(\{x^2\}) = \prod_{j>i}^N (x_j^2 - x_i^2)$ is the Vandermonde determinant. It is normalised to unity

$$1 \;=\; \int_0^\infty \prod_{i=1}^{N} dx_i dy_i \; \mathcal{P}_\nu^{(N_f)}(\{x\},\{y\};\{m_1\},\{m_2\}) \; . \tag{2.8}$$

From the joint probability distribution we can define an $(n,k)$-*density correlation function*

$$\begin{aligned} R_{k,l}(\{x\}_k,\{y\}_l) &\equiv \frac{N!^2}{(N-k)!(N-l)!}\int_0^\infty \prod_{i=k+1}^{N} dx_i \prod_{j=l+1}^{N} dy_j\ \mathcal{P}_\nu^{(N_f)}(\{x\},\{y\};\{m_1\},\{m_2\}) \\ &= \frac{N!^2}{(N-k)!(N-l)!}\frac{1}{\mathcal{Z}_\nu^{(N_f)}} \\ &\quad\times\int_0^\infty \prod_{i=k+1}^{N} dx_i \prod_{j=l+1}^{N} dy_j \det\left[w_\nu^{(N_f)}(x_i,y_j)\right]\Delta_N(\{x^2\})\Delta_N(\{y^2\}), \end{aligned} \tag{2.9}$$

where we have moved the exponential and masses into the determinant, introducing

$$w_\nu^{(N_f)}(x_i,y_j)\ \equiv\ (x_iy_j)^{\nu+1}\ e^{-N(c_1x_i^2+c_2y_j^2)}I_\nu(2dNx_iy_j)\prod_{f1=1}^{N_1}(x_i^2+m_{f1}^2)\prod_{f2=1}^{N_2}(y_j^2+m_{f2}^2)\ . \tag{2.10}$$

Obviously $R_{0,0}=1$ is normalised to unity. The $R_{k,l}(\{x\}_k,\{y\}_l)$ can be expressed in terms of a determinant of four different kernels. These are given by the (bi-)orthogonal polynomials and their integral transforms with respect to the weight function eq. (2.10), and we refer to [12] for details. The $R_{k,l}(\{x\}_k,\{y\}_l)$ will be the building blocks to compute the gap probabilities as well as the distributions of individual eigenvalues of both type $x$ and $y$.

We define the following gap probabilities as

$$\begin{aligned} E_{k,l}(s,t) \equiv \frac{N!^2}{(N-k)!(N-l!)}&\int_0^s dx_1\dots dx_k\int_s^\infty dx_{k+1}\dots dx_N\int_0^t dy_1\dots dy_l\int_t^\infty dy_{l+1}\dots dy_N \\ &\times\ \mathcal{P}_\nu^{(N_f)}(\{x\},\{y\};\{m_1\},\{m_2\})\,,\quad \text{for } k,l=0,1,\dots,N\ , \end{aligned} \tag{2.11}$$

where in the sequel we suppress the dependence on masses and topology for simplicity. The $E_{k,l}(s,t)$ give the probability for general $k,l\in\ \{0,\dots,N\}$ that the interval $[0,s]$ is occupied by $k$ $x$-eigenvalues of $\mathcal{D}_1$ and $[s,\infty)$ is occupied by $(N-k)$ $x$-eigenvalues, and that the interval $[0,t]$ is occupied by $l$ $y$-eigenvalues of $\mathcal{D}_2$ and $[t,\infty)$ is occupied by $(N-l)$ $y$-eigenvalues. It also depends on the masses and on $\mu_{1,2}$ which we have suppressed here.

Similarly we can define the probability to find the $k$-th $x$-eigenvalue at value $x_k=s$, and the $l$-th $y$-eigenvalue at value $y_l=t$, to be

$$\begin{aligned} p_{k,l}(s,t)\equiv k\binom{N}{k}l\binom{N}{l}&\int_0^s dx_1\dots dx_{k-1}\int_s^\infty dx_{k+1}\dots dx_N\int_0^t dy_1\dots dy_{l-1}\int_t^\infty dy_{l+1}\dots dy_N \\ &\times\mathcal{P}_\nu^{(N_f)}(x_1,\dots,x_{k-1},x_k=s,x_{k+1},\dots,x_N,y_1,\dots,y_{l-1},y_l=t,y_{l+1},\dots,y_N;\{m_1\},\{m_2\}). \end{aligned} \tag{2.12}$$

Here the eigenvalues are ordered, $x_1\le\ldots\le x_N$ and $y_1\le\ldots\le y_N$, and obviously $k,l\ge 1$. The fact that the $p_{k,l}(s,t)$ are probabilities that are normalised as

$$\int_0^\infty ds\int_0^\infty dt\ p_{k,l}(s,t)\ =\ 1\ , \tag{2.13}$$

can be seen along the same lines as for a single set of Dirac operator eigenvalues, as was shown in the appendix of [5].

Because we have two sets of eigenvalues we may also define mixed gap-probability distributions as they will occur in intermediate steps. There are two different functions defined as

$$\begin{aligned} Ep_{k,l}(s,t) \equiv & \frac{N!}{(N-k)!} l \binom{N}{l} \int_0^s dx_1 \dots dx_k \int_s^\infty dx_{k+1} \dots dx_N \int_0^t dy_1 \dots dy_{l-1} \int_t^\infty dy_{l+1} \dots dy_N \\ & \times \mathcal{P}_\nu^{(N_f)}(\{x\}, y_1, \dots, y_{l-1}, y_l = t, y_{l+1}, \dots, y_N; \{m_1\}, \{m_2\}) \,, \\ & \text{for } k = 0, \dots, N \;\; \text{and } l = 1, \dots, N \;\; , \end{aligned} \tag{2.14}$$

$$\begin{aligned} pE_{k,l}(s,t) \equiv & k \binom{N}{k} \frac{N!}{(N-l!)} \int_0^s dx_1 \dots dx_{k-1} \int_s^\infty dx_{k+1} \dots dx_N \int_0^t dy_1 \dots dy_l \int_t^\infty dy_{l+1} \dots dy_N \\ & \times \mathcal{P}_\nu^{(N_f)}(x_1, \dots, x_{k-1}, x_k = s, x_{k+1}, \dots, x_N, \{y\}; \{m_1\}, \{m_2\}) \,, \\ & \text{for } k = 1, \dots, N \;\; \text{and } l = 0, \dots, N \;\; . \end{aligned} \tag{2.15}$$

The first quantity eq. (2.14) gives the probability that $[0, s]$ is occupied by $k$ of the $x$-eigenvalues of $\mathcal{D}_1$ and $[s, \infty)$ is occupied by $(N - k)$ of the $x$-eigenvalues, given that $y_l = t$, where the $y$-eigenvalues are ordered. The second quantity eq. (2.15) gives the probability that $[0, t]$ is occupied by $l$ of the $y$-eigenvalues of $\mathcal{D}_2$ and $[t, \infty)$ is occupied by $(N - l)$ of the $y$-eigenvalues, given that $x_k = s$, where again the $x$-eigenvalues are ordered.

These definitions include for example the probability $pE_{k,0}(s, t = 0)$ to find an eigenvalue of the first type at $x = s$, where all $y$-eigenvalues are integrated out. We will return to this in section 4.

## 3. Gap probabilities and individual eigenvalues from densities

We use the simple identity

$$(a-b)^j \;=\; \sum_{l=0}^{j} (-1)^l \binom{j}{l} a^{j-l} b^l \,, \tag{3.1}$$

and choose $a = \int_0^\infty dx$ and $b = \int_0^s dx$ to replace all the $(N - k)$ $dx$-integrals $\int_s^\infty dx$ in eq. (2.11) by $a - b$, and likewise for the corresponding $y$-integrations. We obtain

$$\begin{aligned} E_{k,l}(s,t) = & \frac{N!^2}{(N-k)!(N-l)!} \int_0^s dx_1 \dots dx_k \int_0^t dy_1 \dots dy_l \\ & \times \sum_{i=0}^{N-k} (-1)^i \binom{N-k}{i} \left( \int_0^\infty \right)^{N-k-i} \left( \int_0^s \right)^i dx_{k+1} \dots dx_N \\ & \times \sum_{j=0}^{N-l} (-1)^j \binom{N-l}{j} \left( \int_0^\infty \right)^{N-l-j} \left( \int_0^t \right)^j dy_{l+1} \dots dy_N \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}; \{m_1\}, \{m_2\}) \\ = & \sum_{i=0}^{N-k} \sum_{j=0}^{N-l} \frac{(-1)^{i+j}}{i!j!} \int_0^s dx_1 \dots dx_{k+i} \int_0^t dy_1 \dots dy_{l+j} R_{k+i,l+j}(x_1, \dots, x_{k+i}, y_1, \dots, y_{l+j}). \end{aligned} \tag{3.2}$$

Here we have used the invariance of the joint probability distribution under permutations of both $\{x\}$ and $\{y\}$. The formula (3.2) neatly expresses the gap probability in terms of spectral correlation functions of both sets of eigenvalues. The latter can be derived from $k$-point resolvents in chiral perturbation theory by enlarging eq. (2.1) to the corresponding supergroup integral, see [4]. Thus we have shown how in this setting also gap probabilities and individual eigenvalue distributions follow at the level of the chiral Lagrangian.

We may introduce a generating functional for all gap probabilities,

$$E(s,t;\xi,\eta) \equiv \sum_{i,j=0}^{N} (-\xi)^i (-\eta)^j \frac{1}{i!j!} \int_0^s dx_1 \dots dx_i \int_0^t dx_1 \dots dx_j \, R_{i,j}(x_1,\dots,x_i,\, y_1,\dots,y_j)\,, \tag{3.3}$$

where the term at $i = j = 0$ gives unity. It immediately follows that

$$E_{k,l}(s,t) = (-1)^{k+l} \frac{\partial^k}{\partial \xi^k} \frac{\partial^l}{\partial \eta^l} E(s,t;\xi,\eta)\Big|_{\xi=1,\eta=1}, \quad \text{for } k,l = 0,1,\dots,N \ . \tag{3.4}$$

We will now relate gap probabilities, mixed and individual eigenvalue distributions to density correlations. It can be easily shown that

$$\frac{\partial}{\partial s} E_{k,l}(s,t) = k!\,(pE_{k,l}(s,t) - pE_{k+1,l}(s,t)) \ . \tag{3.5}$$

For $k = l = 0$ we have

$$\frac{\partial}{\partial s} E_{0,0}(s,t) = -pE_{1,0}(s,t)\,, \tag{3.6}$$

as from the definition $pE_{k,l}$ has $k \geq 1$, and thus we set $pE_{0,l}(s,t) \equiv 0$. Similarly it follows

$$\frac{\partial}{\partial t} E_{k,l}(s,t) = l!\,(Ep_{k,l}(s,t) - Ep_{k,l+1}(s,t))\ , \tag{3.7}$$

where again $Ep_{k,0}(s,t) \equiv 0$. If we differentiate the mixed correlators we obtain

$$\frac{\partial}{\partial s} Ep_{k,l}(s,t) = k!\,(p_{k,l}(s,t) - p_{k+1,l}(s,t))\ , \tag{3.8}$$

$$\frac{\partial}{\partial t} pE_{k,l}(s,t) = l!\,(p_{k,l}(s,t) - p_{k,l+1}(s,t)) \ . \tag{3.9}$$

Finally, if we differentiate the gap probabilities twice we arrive at

$$\frac{\partial^2}{\partial s \partial t} E_{k,l}(s,t) = k!\; l!\,(p_{k,l}(s,t) - p_{k+1,l}(s,t) - p_{k,l+1}(s,t) + p_{k+1,l+1}(s,t)) \ . \tag{3.10}$$

Of course the order of differentiation does not matter, as one can easily convince oneself. The boundary conditions to be imposed here and in eqs. (3.8) and (3.9) follow from

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} E_{k,0}(s,t) &= -\,k!\,(p_{k,1}(s,t) - p_{k+1,1}(s,t))\ , \\ \frac{\partial^2}{\partial s \partial t} E_{0,l}(s,t) &= -\,l!\,(p_{1,l}(s,t) - p_{1,l+1}(s,t)) \ . \end{aligned} \tag{3.11}$$

Again from the definitions we have $p_{k,0}(s,t) = p_{0,l}(s,t) \equiv 0$.

The probabilities $p_{k,l}(s,t)$ can be solved for the (mixed) gap probabilities in three different ways. Summing over $k$ in eq. (3.8), or over $l$ in eq. (3.9) we obtain

$$p_{n+1,l}(s,t) = -\sum_{k=0}^{n} \frac{1}{k!}\frac{\partial}{\partial s} E p_{k,l}(s,t)\,,$$
$$p_{k,n+1}(s,t) = -\sum_{l=0}^{n} \frac{1}{l!}\frac{\partial}{\partial t} p E_{k,l}(s,t)\ . \tag{3.12}$$

Alternatively one can sum over both $k$ and $l$ in eq. (3.10) to obtain an expression in terms of gap probabilities alone

$$p_{n+1,q+1}(s,t) \;=\; +\sum_{k=0}^{n}\sum_{l=0}^{q} \frac{1}{k!\,l!}\frac{\partial^2}{\partial s\partial t} E_{k,l}(s,t)\ . \tag{3.13}$$

Let us give some examples. For the simplest case of $k = l = 0$ we get the probability that the interval $[0,s]$ is free of $x$-, and the interval $[0,t]$ free of $y$-eigenvalues:

$$E_{0,0}(s,t) \;=\; \int_s^\infty \prod_{i=1}^{N} dx_i \int_t^\infty \prod_{j=1}^{N} dy_j\ \mathcal{P}_\nu^{(N_f)}(\{x\},\{y\};\{m_1\},\{m_2\})\,, \tag{3.14}$$

and we obtain

$$\frac{\partial^2}{\partial s\partial t} E_{0,0}(s,t) \;=\; p_{1,1}(s,t)\ . \tag{3.15}$$

Explicitly we have for this gap probability the expansion eq. (3.2) given already in [16]

$$\begin{aligned} E_{0,0}(s,t) = 1 &- \int_0^s dx\, R_{1,0}(x) - \int_0^t dy\, R_{0,1}(y) + \frac{1}{2}\int_0^s dx_1 dx_2\, R_{2,0}(x_1,x_2) \\ &+\frac{1}{2}\int_0^t dy_1 dy_2\, R_{0,2}(y_1,y_2) \;+\; \ldots \\ &+\int_0^s dx \int_0^t dy R_{1,1}(x,y) - \frac{1}{2}\int_0^s dx_1 dx_2 \int_0^t dy\, R_{2,1}(x_1,x_2,y) \\ &-\frac{1}{2}\int_0^s dx \int_0^t dy_1 dy_2\, R_{1,2}(x,y_1,y_2) \;+\; \ldots \quad . \end{aligned} \tag{3.16}$$

The terms in the first two lines containing only $s$- or $t$-dependent integrals are annihilated by the differentiation in eq. (3.15), and we get to the same order

$$p_{1,1}(s,t) \;=\; R_{1,1}(s,t) \;-\; \int_0^s dx\, R_{2,1}(x,s,t) \;-\; \int_0^t dy\, R_{1,2}(s,t,y) \;+\; \ldots\ . \tag{3.17}$$

## 4. An exact expression for the first eigenvalue distribution

In this section we derive a closed expression for an individual eigenvalue distribution, the probability to find the first eigenvalue of $\mathcal{D}_1$ at $s$ irrespective of the position of the $\mathcal{D}_2$-eigenvalues. Our solution given in terms of a new kernel and polynomials holds for any number of flavours $N_1$ and $N_2$ and arbitrary chemical potentials $\mu_1$ and $\mu_2$. In particular, we can partially quench the type-1 flavours (putting $N_1 = 0$) with $\mu_1 \neq 0$ in gauge theory with type-2 physical sea-quark flavours (*i.e.*, $N_2 \neq 0$) with $\mu_2 = 0$. This case is probably the most interesting for applications to lattice QCD.

### 4.1 The finite-$N$ solution

We first consider the gap probability that the interval $[0, s]$ is empty of $x$-eigenvalues,

$$E_{0,0}(s, t = 0) = \int_s^\infty dx_1 \dots dx_N \int_0^\infty dy_1 \dots dy_N \; \mathcal{P}_\nu^{(N_f)}(\{x\}, \{y\}; \{m_1\}, \{m_2\}) \; . \tag{4.1}$$

From this the sought probability follows by differentiation, $pE_{1,0}(s, t = 0) = -\partial_s E_{0,0}(s, t = 0)$. We will now perform a series of steps before arriving at an exact expression for finite $N$. The appropriate large-$N$ scaling limit will be taken in the next subsection.

First, recalling the definition of the joint probability distribution eq. (2.7) we can use the fact that all $y$-eigenvalues are integrated out in eq. (4.1) for a symmetry argument. The Vandermonde determinant $\Delta_N(\{y^2\})$ and the determinant of the Bessel function are antisymmetric. Therefore we can replace the latter by its diagonal part times $N!$

$$\begin{aligned} E_{0,0}(s, 0) \; = \; & \frac{N!}{\mathcal{Z}_\nu^{(N_f)}} \int_s^\infty dx_1 \dots dx_N \int_0^\infty dy_1 \dots dy_N \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \\ & \times \prod_{i=1}^N \left( (x_i y_i)^{\nu+1} \; e^{-N(c_1 x_i^2 + c_2 y_i^2)} I_\nu(2dN x_i y_i) \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right). \end{aligned} \tag{4.2}$$

In the next step we use a known identity relating the Laguerre weight times the $I$-Bessel function to an *infinite* sum over Laguerre polynomials (see e.g. eq. (B.7) in [12]). With this decomposition we can exploit the orthogonality properties of these polynomials. For simplicity of the proof we will first consider $N_1 = N_2 = 1$ ($N_f = 2$) with masses $m_1$ and $m_2$, and later give the general result for any numbers of flavours. We thus have

$$\begin{aligned} E_{0,0}(s, 0) = & \frac{N!}{\mathcal{Z}_\nu^{(1+1)}} \int_s^\infty dx_1 \dots dx_N \int_0^\infty dy_1 \dots dy_N \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \prod_{i=1}^N (x_i^2 + m_1^2)(y_i^2 + m_2^2) \\ & \times \prod_{i=1}^N \left( (Nd)^\nu \tau^{\nu+1} (x_i y_i)^{2\nu+1} e^{-N\tau(c_1 x_i^2 + c_2 y_i^2)} \sum_{n_i=0}^\infty \frac{n_i! (1-\tau)^{n_i}}{(n_i + \nu)!} L_{n_i}^\nu(N\tau c_1 x_i^2) L_{n_i}^\nu(N\tau c_2 y_i^2) \right), \end{aligned} \tag{4.3}$$

with $\tau = 1 - d^2/(c_1 c_2)$. Next we include the mass $m_2$ into the Vandermonde determinant $\Delta_N(\{y^2\})$, and then replace it by a determinant of Laguerre polynomials normalised to be monic

$$\Delta_N(\{y^2\}) \prod_{i=1}^N (y_i^2 + m_2^2) = \Delta_{N+1}((im_2)^2, \{y^2\}) = \det_{j,k=0,\dots,N} \left[ (-)^j j! (N\tau c_2)^{-j} L_j^\nu(N\tau c_2 y_k^2) \right] \; , \tag{4.4}$$

where $y_0 \equiv im_2$. We observe that the Laguerre polynomials $L_j^\nu(N\tau c_2 y_k)$ now all appear with their corresponding weight function $y_i^{2\nu+1} e^{-N\tau c_2 y_i^2}$, except for $y_0$ of course.[2] Writing the determinant eq. (4.4) as a sum over permutations we can integrate out all variables $y_1$

[2] Note the additional $\tau$ in the exponent comparing eq. (4.2) and (4.3), coming from the identity for the $I$-Bessel function.

to $y_N$ successively, each integral killing one infinite sum over $n_i$. Thus, under the permutation each $L_{n_i}^{(\nu)}(N\tau c_2 y_i^2)$ gets replaced by $L_{n_i}^{(\nu)}(N\tau c_1 x_i^2)$ times the norm from the integration and the remaining factor from inside the sum. We can therefore rewrite the result again as a determinant, with the first row containing the mass $y_0 = im_2$ unchanged:

$$
\begin{aligned}
E_{0,0}(s,0) = & \frac{N!}{\mathcal{Z}_\nu^{(1+1)}} (Nd)^{N\nu} \tau^{N(\nu+1)} \int_s^\infty dx_1 \dots dx_N \Delta_N(\{x^2\}) \prod_{i=1}^N (x_i^2 + m_1^2)\ x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \\
& \times \begin{vmatrix}
L_0^\nu(N\tau c_2 (im_2)^2) & \cdots & \frac{(-)^j j!}{(N\tau c_2)^j} L_j^\nu(N\tau c_2 (im_2)^2) & \cdots & \frac{(-)^N N!}{(N\tau c_2)^N} L_N^\nu(N\tau c_2 (im_2)^2) \\
\frac{1}{2(N\tau c_2)^{\nu+1}} L_0^\nu(N\tau c_1 x_1^2) & \cdots & \frac{(-)^j j!(1-\tau)^j}{2(N\tau c_2)^{j+\nu+1}} L_j^{(\nu)}(N\tau c_1 x_1^2) & \cdots & \frac{(-)^N N!(1-\tau)^N}{2(N\tau c_2)^{N+\nu+1}} L_N^{(\nu)}(N\tau c_1 x_1^2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{2(N\tau c_2)^{\nu+1}} L_0^\nu(N\tau c_1 x_N^2) & \cdots & \frac{(-)^j j!(1-\tau)^j}{2(N\tau c_2)^{j+\nu+1}} L_j^{(\nu)}(N\tau c_1 x_N^2) & \cdots & \frac{(-)^N N!(1-\tau)^N}{2(N\tau c_2)^{N+\nu+1}} L_N^{(\nu)}(N\tau c_1 x_N^2)
\end{vmatrix} \\
= & \frac{N!(Nd)^{N\nu}\tau^{N(\nu+1)} \prod_{j=0}^N (1-\tau)^j (N\tau c_2)^{-j}}{\mathcal{Z}_\nu^{(1+1)} 2^N (N\tau c_2)^{N(\nu+1)}} \int_s^\infty dx_1 \dots dx_N \Delta_N(\{x^2\}) \prod_{i=1}^N (x_i^2 + m_1^2) x_i^{2\nu+1} e^{-N\tau c_1 x_i^2} \\
& \times \begin{vmatrix}
\hat{L}_0^\nu(N\tau c_2 (im_2)^2) & \cdots & (1-\tau)^{-j} \hat{L}_j^\nu(N\tau c_2 (im_2)^2) & \cdots & (1-\tau)^{-N} \hat{L}_N^\nu(N\tau c_2 (im_2)^2) \\
\hat{L}_0^\nu(N\tau c_1 x_1^2) & \cdots & \hat{L}_j^\nu(N\tau c_1 x_1^2) & \cdots & \hat{L}_N^\nu(N\tau c_1 x_1^2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hat{L}_0^\nu(N\tau c_1 x_N^2) & \cdots & \hat{L}_j^\nu(N\tau c_1 x_N^2) & \cdots & \hat{L}_N^\nu(N\tau c_1 x_N^2)
\end{vmatrix}
\end{aligned}
\tag{4.5}
$$

taking out common factors. Here we have defined the following notation for monic Laguerre polynomials

$$
\hat{L}_n^\nu(x) \equiv (-1)^n n!\ L_n^\nu(x) \ = \ \sum_{j=0}^n (-1)^{n+j} \frac{n!(n+\nu)!}{(n-j)!(\nu+j)!j!}\ x^j \quad . \tag{4.6}
$$

The last determinant in eq. (4.5) can be almost mapped to a Vandermonde determinant, using the following identity:

$$
\begin{aligned}
& \begin{vmatrix}
\hat{L}_0^\nu(M_2^2) & \cdots & \frac{1}{(1-\tau)^j} \hat{L}_j^\nu(M_2^2) & \cdots & \frac{1}{(1-\tau)^N} \hat{L}_N^\nu(M_2^2) \\
\hat{L}_0^\nu(X_1^2) & \cdots & \hat{L}_j^\nu(X_1^2) & \cdots & \hat{L}_N^\nu(X_1^2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\hat{L}_0^\nu(X_N^2) & \cdots & \hat{L}_j^\nu(X_N^2) & \cdots & \hat{L}_N^\nu(X_N^2)
\end{vmatrix} = \\
& \qquad = \begin{vmatrix}
\hat{L}_0^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^j}{(1-\tau)^j} \hat{L}_j^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^N}{(1-\tau)^N} \hat{L}_N^\nu(\frac{M_2^2}{\tau}) \\
1 & \cdots & X_1^{2j} & \cdots & X_1^{2N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & X_N^{2j} & \cdots & X_N^{2N}
\end{vmatrix},
\end{aligned}
\tag{4.7}
$$

where we have defined

$$
M_2^2 \equiv N\tau c_2 (im_2)^2 \ \text{ and } \ X_k^2 \equiv N\tau c_1 x_k^2 \ . \tag{4.8}
$$

A proof of this relation is given in appendix A.



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We can now change variables $x_j \to u_j = x_j^2$, and perform the shift $u_j \to z_j = u_j - s^2$ to obtain integrations $\int_0^\infty dz_j$ in eq. (4.5). The change of variables and subsequent shifts induce the following changes:

$$\begin{aligned}
\Delta_N(\{u\}) &\to \Delta_N(\{z\})\,, \\
(u_j + m_1^2) &\to (z_j + s^2 + m_1^2) \equiv (z_j + m_1'^2)\,, \\
u_j^\nu &\to (z_j + s^2)^\nu\,, \\
e^{-N\tau c_1 u_i} &\to e^{-N\tau c_1 z_i} e^{-N\tau c_1 s^2}\ .
\end{aligned} \tag{4.9}$$

In other words: the Vandermonde determinant remains invariant, the mass $m_1$ receives a shift to $m_1'^2 = s^2 + m_1^2$, the topology term becomes a $\nu$-fold degenerate mass term with mass $s^2$, and the weight is shifted by a constant factor. While this is just as in the chiral one-matrix theory, the difference here is that the almost Vandermonde eq. (4.7) is not invariant, and becomes

$$\begin{aligned}
&\begin{vmatrix}
\hat{L}_0^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^j}{(1-\tau)^j}\hat{L}_j^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^N}{(1-\tau)^N}\hat{L}_N^\nu(\frac{M_2^2}{\tau}) \\
1 & \cdots & X_1^{2j} & \cdots & X_1^{2N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & X_N^{2j} & \cdots & X_N^{2N}
\end{vmatrix} = \\
&= \begin{vmatrix}
\hat{L}_0^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^j}{(1-\tau)^j}\hat{L}_j^\nu(\frac{M_2^2}{\tau}) & \cdots & \frac{\tau^N}{(1-\tau)^N}\hat{L}_N^\nu(\frac{M_2^2}{\tau}) \\
1 & \cdots & (N\tau c_1(z_1+s^2))^j & \cdots & (N\tau c_1(z_1+s^2))^N \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & (N\tau c_1(z_N+s^2))^j & \cdots & (N\tau c_1(z_N+s^2))^N
\end{vmatrix} \\
&= \begin{vmatrix}
\hat{L}_0^\nu(\frac{M_2^2}{\tau}) & \cdots & \sum_{l=0}^j \frac{\tau^l}{(1-\tau)^l}\hat{L}_l^\nu(\frac{M_2^2}{\tau})(-S^2)^{j-l}\binom{j}{l} & \cdots & \sum_{l=0}^N \frac{\tau^l}{(1-\tau)^l}\hat{L}_l^\nu(\frac{M_2^2}{\tau})(-S^2)^{N-l}\binom{N}{l} \\
1 & \cdots & Z_1^j & \cdots & Z_1^N \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & Z_N^j & \cdots & Z_N^N
\end{vmatrix},
\end{aligned} \tag{4.10}$$

where we have defined

$$S^2 \equiv N\tau c_1 s^2 \quad \text{and} \quad Z_k \equiv N\tau c_1 z_k\ . \tag{4.11}$$

Here we have subsequently added columns[3] to turn back to monic powers in $Z_k$. The fact that it is not quite invariant illustrates the fundamental property that there is apparently no way to map the present two-matrix problem onto an equivalent one-matrix problem.

In the next step we turn the monic powers $Z_i^j$ back into monic Laguerre polynomials, using again the invariance of the determinant. This will introduce yet another sum over the Laguerre polynomials in the mass variable $M_2^2$ in the first row. Because of the shift eq. (4.9) we only need to generate Laguerre polynomials with topology $\nu = 0$, $\hat{L}_j^{\nu=0} \equiv \hat{L}_j$, to obtain the polynomials orthogonal to the shifted weight. We thus obtain for the last

[3]Usually one goes from monic powers to monic polynomials in this step. It is easy to invert this by defining $Z_i' \equiv Z_i + S^2$ and then going from monic powers $Z_i'^j$ to polynomials $(Z_i' - S^2)^j = Z_i^j$.

determinant in eq. (4.10)

$$\begin{vmatrix} Q_0(M_2^2) & \cdots & Q_j(M_2^2) & \cdots & Q_N(M_2^2) \\ \hat{L}_0(Z_1) & \cdots & \hat{L}_j(Z_1) & \cdots & \hat{L}_N(Z_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{L}_0(Z_N) & \cdots & \hat{L}_j(Z_N) & \cdots & \hat{L}_N(Z_N) \end{vmatrix}, \tag{4.12}$$

which defines *new polynomials*

$$\begin{aligned} Q_n(M_2^2) &\equiv \sum_{j=0}^{n} \frac{(-)^{n+j}(n!)^2}{(n-j)!(j!)^2} \sum_{l=0}^{j} \frac{\tau^l}{(1-\tau)^l} \hat{L}_l\left(\frac{M_2^2}{\tau}\right) (-S^2)^{j-l} \binom{j}{l} \\ &= (-)^n n! \sum_{l=0}^{n} \frac{\tau^l}{(1-\tau)^l} L_l\left(\frac{M_2^2}{\tau}\right) L_{n-l}^l(-S^2) \\ &= (-)^n n! \sum_{l=0}^{n} L_{n-l}^l(-S^2) \sum_{k=0}^{l} \frac{(-)^{k+l} l!}{(l-k)!k!(1-\tau)^k} L_k(M_2^2) \\ &= (-)^n n! \sum_{k=0}^{n} \frac{1}{(1-\tau)^k} L_k(M_2^2) L_{n-k}^{-1}(-S^2) \ . \end{aligned} \tag{4.13}$$

In the first step we have swapped sums, $\sum_{j=0}^{n}\sum_{l=0}^{j} \to \sum_{l=0}^{n}\sum_{j=l}^{n}$, such that the powers in $S^2$ give the Laguerre polynomial $L_{n-l}^l(-S^2)$. In the second step we have used the identity (A.8) backwards in order to take the argument $1/\tau$ out of the first Laguerre polynomial, in choosing $z = M_2^2$ and $w = 1/\tau$ in eq. (A.8). This goes at the expense of introducing another sum. After swapping again sums to $\sum_{k=0}^{n}\sum_{l=k}^{n}$, the latter sum over the generalised Laguerre polynomial in $-S^2$ can be simplified, using the following identity,

$$L_{n-k}^{-1}(-S^2) \ = \ \sum_{j=0}^{n-k} (-)^j \binom{j+k}{k} L_{n-k-j}^{j+k}(-S^2) \ . \tag{4.14}$$

A proof of this simple identity is presented in the appendix B. Note that all polynomials $L_{n-k}^{-1}(-S^2)$ are proportional to $-S^2$, except for $n = k$ as $L_0^{-1}(-S^2) = 1$. This will become important when computing the normalisation in the limit $s \to 0$.

The explicit appearance of a new set of polynomials is again a reminder that we cannot map the problem onto a one-matrix problem. Of course, in the limit of the deformation parameters $\mu_{1,2} \to 0$, the polynomials $Q_n(M_2^2)$ reduce to Laguerre polynomials. In the form given in the last line of eq. (4.13) the new polynomials $Q_n$ are amenable to the large-$N$ scaling limit that we take in the next subsection.

Returning to the gap probability, in the last step we now replace in eq. (4.5) the determinant $\Delta_N(\{x^2\}) = \Delta_N(\{z\})$ times the mass term by a larger Vandermonde determinant, and then replace monic powers by Laguerre polynomials monic in the arguments $z_k$. The $\nu$ degenerate masses obtained after the shift eq. (4.9) can be dealt with by first taking them different, and then taking limits by multiple application of l'Hôpital's rule. For simplicity

we set $\nu = 0$ in all of the following. We have

$$\Delta_N(\{z\})\prod_{i=1}^{N}(z_i + m_1'^2) = \Delta_{N+1}((im_1')^2, \{z\}) = \det_{j,k=0,\ldots,N}\left[(-)^j j!(N\tau c_1)^{-j} L_j(N\tau c_1 z_k)\right] , \tag{4.15}$$

where we define $z_0 = im_1'$.

Let us collect what we have derived so far:

$$\begin{aligned} E_{0,0}(s,0) = & \frac{N!\prod_{j=0}^{N}(1-\tau)^j(N\tau c_2)^{-j}(N\tau c_1)^{-j}}{\mathcal{Z}_0^{(1+1)}2^{2N}(Nc_2)^N} e^{-N^2\tau c_1 s^2}\int_0^\infty dz_1\ldots dz_N\ e^{-N\sum_{i=0}^{N}\tau c_1 z_i} \\ & \times \begin{vmatrix} \hat{L}_0(M_1'^2) & \cdots & \hat{L}_j(M_1'^2) & \cdots & \hat{L}_N(M_1'^2) \\ \hat{L}_0(Z_1) & \cdots & \hat{L}_j(Z_1) & \cdots & \hat{L}_N(Z_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{L}_0(Z_N) & \cdots & \hat{L}_j(Z_N) & \cdots & \hat{L}_N(Z_N) \end{vmatrix} \left| \begin{matrix} Q_0(M_2^2) & \cdots & Q_j(M_2^2) & \cdots & Q_N(M_2^2) \\ \hat{L}_0(Z_1) & \cdots & \hat{L}_j(Z_1) & \cdots & \hat{L}_N(Z_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{L}_0(Z_N) & \cdots & \hat{L}_j(Z_N) & \cdots & \hat{L}_N(Z_N) \end{matrix} \right| . \end{aligned} \tag{4.16}$$

Here the definition

$$M_1'^2 \equiv N\tau c_1(im_1')^2 = -N\tau c_1(m_1^2 + s^2) \tag{4.17}$$

has been used. We can now apply the orthogonality of the Laguerre polynomials with respect to the weight $e^{-N\tau c_1 z_i}$ to compute the $N$-fold integral over the determinants, applying the standard Dyson Theorem. We thus obtain the final answer for finite $N$:

$$\begin{aligned} E_{0,0}(s,0) &= C\, e^{-N^2\tau c_1 s^2} K_{N+1}(M_1'^2, M_2^2)\,, \\ K_{N+1}(M_1'^2, M_2^2) &\equiv \sum_{j=0}^{N}\frac{(-)^j}{j!}L_j(M_1'^2)Q_j(M_2^2)\ . \end{aligned} \tag{4.18}$$

This result defines a *new kernel* of the polynomials $L_j$ and $Q_j$ in the (shifted) masses. The constant $C \equiv 1/\sum_{j=0}^{N}\frac{1}{(1-\tau)^j}L_j(M_1^2)L_j(M_2^2)$ that is inversely proportional to the partition function $\mathcal{Z}_0^{(1+1)}$ ensures the correct normalisation $\lim_{s\to 0} E_{0,0}(s,0) = 1$. It can be obtained independently by computing the partition function

$$\mathcal{Z}_0^{(1+1)} \;=\; \frac{N!^2\prod_{j=0}^{N}(j!)^2(1-\tau)^j(N\tau c_1)^{-j}(N\tau c_2)^{-j}}{2^{2N}(Nc_1)^N(N\tau c_2)^N}\sum_{j=0}^{N}\frac{1}{(1-\tau)^j}L_j(M_1^2)L_j(M_2^2)\,, \tag{4.19}$$

following the same steps as before but setting $s = 0$. The calculation simplifies in eq. (4.10) so that the $Q_j$'s become Laguerre polynomials. Indeed as a check we can take

$$\lim_{s\to 0} Q_n(M_2^2) \;=\; \frac{(-)^n n!}{(1-\tau)^n}L_n(M_2^2)\,, \tag{4.20}$$

where only the term $L^{-1}_{n-k=0}(-S^2) = 1$ contributes to the sum. This already indicates that this last term in the sum is special.

As a further independent check we may take the limit $\mu_1, \mu_2 \to 0$ ($\Rightarrow \tau \to 0$). In this limit the two Dirac operators become equal, $\mathcal{D}_1 = \mathcal{D}_2$, and we should recover the known one-matrix theory result [20]. Indeed, we get

$$\begin{aligned}\lim_{\mu_1,\,\mu_2\to 0} \frac{(-)^n}{n!} Q_n(M_2^2) &= \sum_{k=0}^{n} L_k(-Nm_2^2) L_{n-k}^{-1}(-Ns^2) \\ &= \sum_{j=0}^{n} \frac{(-)^j n!}{(n-j)!(j!)^2} \sum_{l=0}^{j} M_2^{2l} (-S^2)^{j-l} \binom{j}{l} \;=\; L_n\Big(-N(m_2^2+s^2)\Big)\,,\end{aligned} \tag{4.21}$$

*i.e.*, Laguerre polynomials of shifted mass just as for the first flavour $m_1'$. This follows from the first and last line of the definition eq. (4.13). Inserted into the kernel eq. (4.18) we obtain the one-matrix theory result for the gap probability in terms of the partition function of 2 flavours with shifted masses.

It is straightforward to see that for more flavours, $N_1 > 1$ and $N_2 > 1$, the very same steps still go through (see also the corresponding determinant identity in appendix A). The only difference is that there will be more rows with masses of flavour $N_1$ in the determinant eq. (4.15), and more masses of flavour $N_2$ in the new polynomials $Q_j$ in eq. (4.12). The absorption of the mass terms into a larger Vandermonde determinant leads to inverse Vandermonde determinants in each of the $N_1$ and $N_2$ masses, which can be taken out of the integral. We arrive at

$$\begin{aligned}E_{0,0}(s,0) \sim & \frac{1}{\mathcal{Z}_0^{(N_f)} \Delta_{N_1}(m_{f1}^2)\Delta_{N_2}(m_{f2}^2)} e^{-N^2\tau c_1 s^2} \int_0^\infty dz_1 \dots dz_N \; e^{-N\sum_{i=0}^{N}\tau c_1 z_i} \\ & \times \begin{vmatrix} \hat{L}_0(M_{f1=1}'^2) & \cdots & \hat{L}_{N+N_1-1}(M_{f1=1}'^2) \\ \cdots & \cdots & \cdots \\ \hat{L}_0(M_{N_1}'^2) & \cdots & \hat{L}_{N+N_1-1}(M_{N_1}'^2) \\ \hat{L}_0(Z_1) & \cdots & \hat{L}_{N+N_1-1}(Z_1) \\ \cdots & \cdots & \cdots \\ \hat{L}_0(Z_N) & \cdots & \hat{L}_{N+N_1-1}(Z_N) \end{vmatrix} \begin{vmatrix} Q_0(M_{f2=1}^2) & \cdots & Q_{N+N_2-1}(M_{f2=1}^2) \\ \cdots & \cdots & \cdots \\ Q_0(M_{N_2}^2) & \cdots & Q_{N+N_2-1}(M_{N_2}^2) \\ \hat{L}_0(Z_1) & \cdots & \hat{L}_{N+N_2-1}(Z_1) \\ \cdots & \cdots & \cdots \\ \hat{L}_0(Z_N) & \cdots & \hat{L}_{N+N_2-1}(Z_N) \end{vmatrix}.\end{aligned} \tag{4.22}$$

The orthogonality of Laguerre polynomials can be exploited in the manner of ref. [19]. This leads to the following determinant expressions. For an equal number of flavours $N_1 = N_2$ we have[4]

$$E_{0,0}(s,0) = const. \frac{1}{\mathcal{Z}_0^{(N_f)} \Delta_{N_1}(\{m_{f1}^2\})\Delta_{N_2}(\{m_{f2}^2\})} e^{-N^2\tau c_1 s^2} \det_{1\le f1,f2\le N_1} \big[K_{N+N_1-1}(M_{f1}'^2, M_{f2}^2)\big]\,. \tag{4.23}$$

In the case where $N_1(N_2)$ is larger, the determinant is of size $N_1(N_2)$ and contains additional polynomials $Lj(M_{j1}'^2)$ ($Q_j(M_{k2}^2)$) to fill up the additional columns (rows) [19],

$$\begin{aligned}N_1 > N_2: \;\; E_{0,0}(s,0) = const. & \frac{1}{\mathcal{Z}_0^{(N_f)} \Delta_{N_1}(\{m_{f1}^2\})\Delta_{N_2}(\{m_{f2}^2\})} \; e^{-N^2\tau c_1 s^2} \\ & \times \det_{f1,f2} \big[K_{N+N_2-1}(M_{f1}'^2, M_{f2=1}^2) \cdots K_{N+N_2-1}(M_{f1}'^2, M_{N_2}^2)\, L_{N+N_2}(M_{f1}'^2) \cdots L_{N+N_1-1}(M_{f1}'^2)\big]\end{aligned} \tag{4.24}$$

[4]We omit all mass dependent normalisation constants here that can be obtained easily. In particular they will cancel the Vandermonde determinants of the masses, see e.g. [12] for the partition functions.



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where we display the $f1$-th row in mass $M'^2_{f1}$ of flavour $N_1$, and

$$N_2 > N_1 : \quad E_{0,0}(s,0) = const. \frac{1}{\mathcal{Z}_0^{(N_f)} \Delta_{N_1}(\{m_{f1}^2\}) \Delta_{N_2}(\{m_{f2}^2\})} \, e^{-N^2 \tau c_1 s^2} \tag{4.25}$$
$$\times \det_{f2,f1} \left[ K_{N+N_1-1}(M_{f2}^2, M'^2_{f1=1}) \cdots K_{N+N_1-1}(M_{f2}^2, M'^2_{N_1}) \, Q_{N+N_1}(M_{f2}^2) \cdots Q_{N+N_2-1}(M_{f2}^2) \right]$$

Here we have transposed the matrix to display the $f2$-th row in mass $M_{f2}^2$ of flavour $N_2$.

For example, this includes in particular the interesting case of quenching the first flavour (*i.e.* putting $N_1 = 0$) while keeping its chemical potential nonzero, $\mu_1 \neq 0$. This quenched flavour can then be measured in the background of $N_2 = 2$ flavours with masses $m_1$ and $m_2$ ,

$$N_2 = 2, N_1 = 0 : \; E_{0,0}(s,0) = const. \frac{e^{-N^2 \tau c_1 s^2}}{\mathcal{Z}_0^{(2)}(m_2^2 - m_1^2)} \det \begin{bmatrix} Q_N(-N\tau c_2 m_1^2) & Q_{N+1}(-N\tau c_2 m_1^2) \\ Q_N(-N\tau c_2 m_2^2) & Q_{N+1}(-N\tau c_2 m_2^2) \end{bmatrix}. \tag{4.26}$$

In particular, setting the chemical potential of the dynamical flavours $N_2$ to zero, $\mu_2 = 0$, will not eliminate the other chemical potential $\mu_1 \neq 0$ (see eq. (2.6)), or reduce to a known one-matrix quantity. This non-trivial $\mu$-dependence due to the valence quarks can serve as a clean way to measure the pion decay constant $F_\pi$ from gauge field ensembles generated with dynamical light quarks that carry no chemical potential.

Finally, as was pointed out earlier, the probability corresponding to non-vanishing gauge field topology $\nu \neq 0$ can be introduced by adding $\nu$ extra masses of $\mathcal{D}_1$, and then taking them to be degenerate with value $s^2$.

### 4.2 The large-$N$ limit

In this subsection we take the large-$N$ scaling limit by the same rescaling as in ref. [12], to which we refer for more details. We first derive the limits of all building blocks needed for the general case, and then specify the fully explicit result in three examples in subsequent subsections.

All eigenvalues, the gap and the masses are rescaled in the same way (as would be $\hat{y} = 2Ny$), the usual microscopic limit

$$\begin{aligned} \hat{x} &\equiv 2Nx \,, & \hat{s} &\equiv 2Ns \,, & & \\ \hat{m}_f &\equiv 2Nm_f \,, & \hat{\mu}_f &\equiv 2N\mu_f^2 & \text{for} \qquad & f = 1,2 \,, \\ \hat{\delta} &\equiv \hat{\mu}_2 - \hat{\mu}_1 \,. & & & & \end{aligned} \tag{4.27}$$

All scalings including the chemical potential keeping $N\mu_f^2$ fixed can be read off from the chiral Lagrangian eq. (2.1). For the various constants containing the $\mu_f$ this implies the following scaling:

$$\begin{aligned} \lim_{N,j,k\to\infty} (1-\tau)^{-k} &= \exp\left[\frac{1}{2} r t \hat{\delta}^2\right] \qquad \text{where} \qquad t \equiv j/N, \qquad r \equiv k/j \,, \\ \lim_{N\to\infty} \tau c_f &= 1 \qquad\qquad\qquad\quad \text{for} \qquad f = 1,2 \,. \end{aligned} \tag{4.28}$$

For the Laguerre polynomials the following scaling holds:

$$\lim_{N,j\to\infty} L_j(M_1^{'2} = -N\tau c_1 m_1^{'2}) = I_0(\sqrt{t}\ \hat{m}_1')\,, \tag{4.29}$$

$$\lim_{N,j,k\to\infty} L^{-1}_{j-k\neq 0}(-S^2 = -N\tau c_1 s^2) = \frac{1}{2j}\sqrt{\frac{t}{(1-r)}}\ \hat{s}\ I_1(\sqrt{(1-r)t}\ \hat{s}),$$

recalling $\hat{m}_1^{'2} = \hat{m}_1^2 + \hat{s}^2$. Special care has to be taken in the asymptotic of the new polynomial,

$$\begin{aligned}\lim_{j\to\infty}\frac{(-)^j}{j!}Q_j(M_2^2 = -N\tau c_2 m_2^2) = \lim_{j\to\infty}\left(\sum_{k=0}^{j-1}\frac{1}{(1-\tau)^k}L_k(M_2^2)\ L^{-1}_{j-k}(-S^2) + \frac{1}{(1-\tau)^j}L_j(M_2^2)\cdot 1\right)\\ \Leftrightarrow \quad Q_S(\hat{m}_2;t) \equiv \frac{1}{2}\int_0^1 dr e^{\frac{1}{2}rt\hat{\delta}^2} I_0\left(\sqrt{rt}\hat{m}_2\right)\sqrt{\frac{t}{1-r}}\hat{s}I_1\left(\sqrt{(1-r)t}\hat{s}\right)\\ + e^{\frac{1}{2}t\hat{\delta}^2} I_0\left(\sqrt{t}\ \hat{m}_2\right).\end{aligned} \tag{4.30}$$

Here we have to split off the $s$-independent part $L_0^{-1} = 1$, which is the single term surviving in the limit $s \to 0$, and which hence ensures the normalisability of the probability in that limit. Usually neglecting a single term when replacing a sum by an integral amounts to removing a quantity of measure zero, which should be irrelevant. However, in our case this is not true as the convergence in $\hat{s}$ is not uniform. Therefore we have to treat that term separately and find the "anomalous" $I_0$-term in the scaling limit. That this procedure is correct is checked by computing the normalisation before and after taking the large-$N$ limit. This curious phenomenon together with the appearance of Laguerre polynomials $L^{-1}_{j-k}$ leads to our new microscopic kernel.

The final answer for the microscopic limit of the new kernel in eq. (4.18) thus reads

$$\begin{aligned}K_S(\hat{m}_1', \hat{m}_2) &\equiv \lim_{N\to\infty}\frac{1}{N}K_{N+1}(M_1^{'2}, M_2^2) \\ &= 2\int_0^1 dT T^2 I_0(T\hat{m}_1')\int_0^1 dR R\frac{1}{\sqrt{1-R^2}}e^{\frac{1}{2}R^2T^2\hat{\delta}^2} I_0(RT\hat{m}_2)\hat{s}I_1\left(\hat{s}T\sqrt{1-R^2}\right)\\ &\quad +2\int_0^1 dT T e^{\frac{1}{2}T^2\hat{\delta}^2} I_0(T\hat{m}_1')I_0(T\hat{m}_2)\,,\end{aligned} \tag{4.31}$$

where we have changed to squared variables. This kernel can no longer be related to single partition functions of shifted masses, as it was the case in the one-matrix theory [20].

Likewise we obtain for the normalisation constant which is proportional to the partition function,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N}\frac{1}{(1-\tau)^j}L_j(M_1^2)L_j(M_2^2) \ = \ 2\int_0^1 dT T\exp\left[\frac{1}{2}T^2\hat{\delta}^2\right] I_0(T\hat{m}_1)I_0(T\hat{m}_2)\ . \tag{4.32}$$

Note that the first mass $\hat{m}_1$ is *not* shifted here, in contrast to the previous equation. Partition functions of more flavours follow easily given the building blocks above, together with the general expressions given in [19, 12]. We now have all ingredients to obtain all



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gap probabilities with any flavour content by inserting the asymptotic kernel eq. (4.31) and asymptotic polynomials eqs. (4.29) and (4.30) into the respective eqs. (4.23) – (4.25), normalised by the corresponding partition function.

In the following we give three simple examples that illustrate these very general expressions. In order to guide the eye we mostly display the distribution of the first eigenvalues versus the corresponding eigenvalue density it has to follow. For comparison to Lattice results the gap probability that we give explicitly may be even more useful as it allows for a binning independent comparison with data.

### 4.3 Two light flavours

Let us first consider the gap probability corresponding to two flavours $N_1 = N_2 = 1$, as given in eq. (4.18). Collecting the formulae from above we obtain

$$\begin{aligned}\lim_{N\to\infty} E_{0,0}(s,0) \equiv E_{S\ 0,0}^{(1+1)}(\hat{s},0) = &\left(\int_0^1 dTTe^{\frac{1}{2}T^2\hat{\delta}^2} I_0(T\hat{m}_1)I_0(T\hat{m}_2)\right)^{-1} \exp\left[-\frac{1}{4}\hat{s}^2\right] \\ &\times\left(\int_0^1 dTT^2 I_0(T\hat{m}_1')\int_0^1 dRR\frac{1}{\sqrt{1-R^2}}e^{\frac{1}{2}R^2T^2\hat{\delta}^2} I_0(RT\hat{m}_2)\hat{s}\ I_1\left(\hat{s}T\sqrt{1-R^2}\right)\right. \\ &\left.+\int_0^1 dTTe^{\frac{1}{2}T^2\hat{\delta}^2} I_0(T\hat{m}_1')I_0(T\hat{m}_2)\right). \end{aligned} \tag{4.33}$$

Once more we can perform an analytic check by taking $\hat{\delta} \to 0$ in order to go back to the known one-matrix quantity [20]. Using the following so-called Sonine integral identity [21] (that also follows from the large-$N$ limit of identity eq. (4.21))

$$s\int_0^1 dx\frac{x}{\sqrt{1-x^2}}\ I_0(mx)I_1(s\sqrt{1-x^2})\ +\ I_0(m)\ =\ I_0(\sqrt{m^2+s^2})\,, \tag{4.34}$$

we obtain the known gap probability [20] as a ratio of a two-flavour partition function with shifted masses over one with unshifted ones.[5] As an illustration, we show the distribution of the first eigenvalues $p_{S\ 1,0}^{N_f=1+1}(\hat{s},0) = -\partial_{\hat{s}}E_{S\ 0,0}^{N_f=1+1}(\hat{s},0)$ for different values of $\hat{\delta}$ and compare it to the corresponding densities in figure 1. From [12] we have for the density

$$\rho_{1,0}^{(1+1)}(\hat{x}) = \rho_{1MM}^{Q}(\hat{x}) - \hat{x}\ \frac{\int_0^1 dttJ_0(t\hat{x})I_0(t\hat{m}_1)\int_0^1 dtt\ e^{\frac{1}{2}t^2\hat{\delta}^2}J_0(t\hat{x})I_0(t\hat{m}_2)}{\int_0^1 dtte^{\frac{1}{2}t^2\hat{\delta}^2}I_0(t\hat{m}_1)I_0(t\hat{m}_2)}\,, \tag{4.35}$$

where we have introduced the one-matrix model quenched density $\rho_{1MM}^{Q}(\hat{x})$ from eq. (4.37) below.

At $\hat{\delta} = 0$ eq. (4.35) coincides with the corresponding one-matrix model density eq. (4.38) below. For $\hat{\delta} = 1$ the curve is still close to this density, compare to figure 2. For $\hat{\delta} \gg 1$ the curves approach the one-matrix quantities of *one flavour* $N_1 = 1$ with mass $\hat{m}_1 = 3$ (the flavour corresponding to the $y$-eigenvalues gets quenched), compare again to figure 2 below. This fact can be seen analytically, by taking the limit $\hat{\delta} \to \infty$ in eq. (4.35)

[5] The remaining integral is elementary and gives a $2 \times 2$ determinant of Bessel functions.



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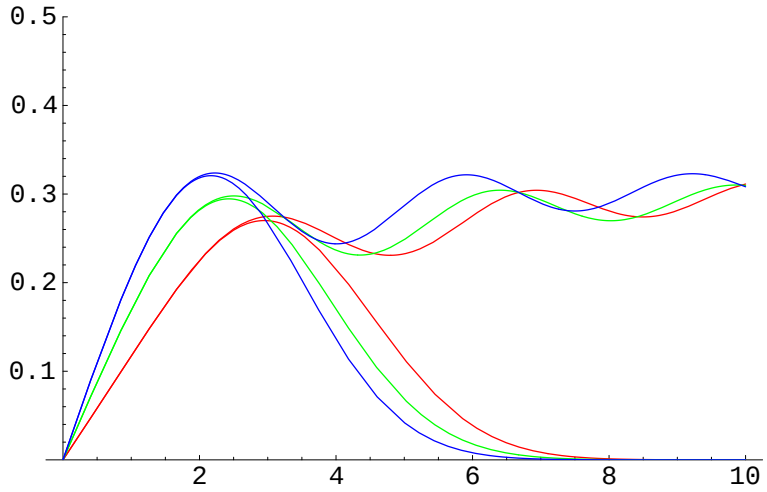


**Figure 1:** The eigenvalue density and first eigenvalue for $N_f = 1 + 1$ with imaginary chemical potential $\hat{\delta} = 1$ (low red), 3 (middle green), and 10 (upper blue curve), at fixed quark masses $\hat{m}_1 = 3$, $\hat{m}_2 = 4$.

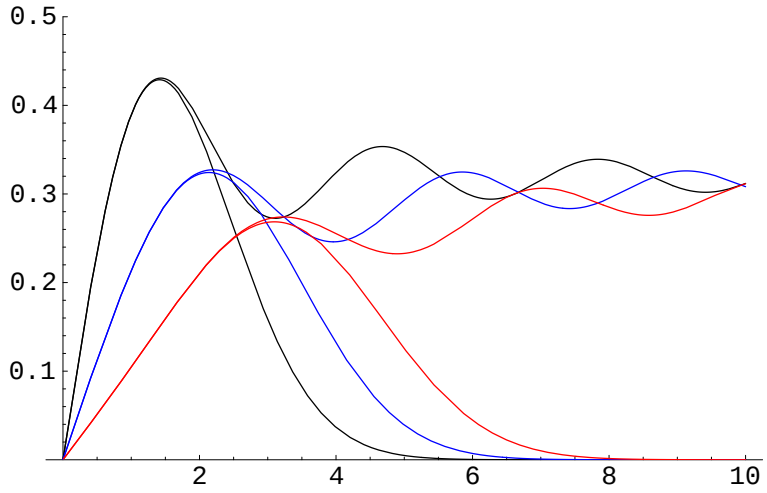


**Figure 2:** The eigenvalue density and first eigenvalue of the one-matrix theory: two flavours with $\hat{m}_1 = 3$, $\hat{m}_2 = 4$ (low red), one flavour with $\hat{m}_1 = 3$ (middle blue), and the quenched case (upper black curve).

and doing a saddle point approximation,

$$\lim_{\hat{\delta}\to\infty} \int_0^1 dtt\ e^{\frac{1}{2}t^2\hat{\delta}^2} J_0(t\hat{x}) I_0(t\hat{m}_2) \ \sim\ \hat{\delta}^{-2} e^{\frac{1}{2}\hat{\delta}^2} J_0(\hat{x}) I_0(\hat{m}_2)\ . \tag{4.36}$$

The $\hat{\delta}$-dependent integrals get replaced by their values at the upper limit $t = 1$, and we have also computed the subleading coefficient for later convenience. After cancelling common factors in eq. (4.35) we obtain, as we should, the one-matrix density for one flavour with mass $\hat{m}_1$ as in eq. (4.40) below. We have checked that the same limit applies to the first eigenvalue distribution.



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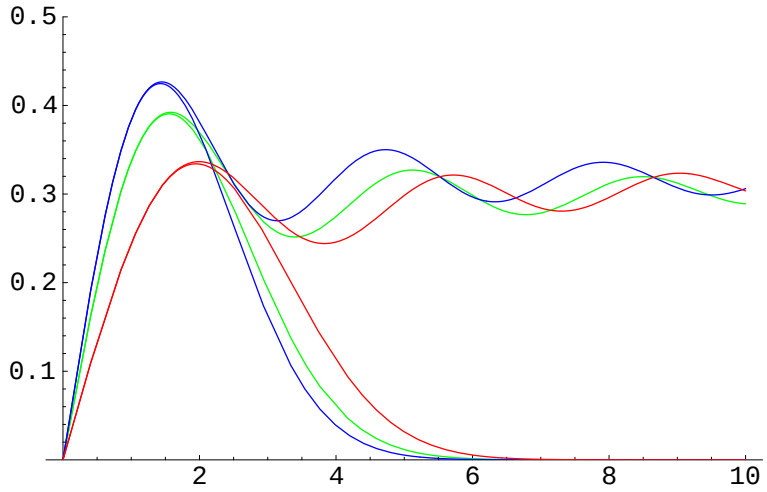


**Figure 3:** The eigenvalue density and first eigenvalue for $N_f = 0 + 1$ with imaginary chemical potential $\hat{\delta} = 1$ (low red), 3 (middle green), and 10 (upper blue curve) and fixed quark mass $\hat{m}_1 = 3$.

For the comparison above we give the following known one-matrix quantities [20, 22] that are displayed in figure 2. The quenched density and its first eigenvalues read

$$\rho^{Q}_{1MM}(\hat{x}) \;=\; \frac{\hat{x}}{2}\left(J_0(\hat{x})^2 + J_1(\hat{x})^2\right)\,, \qquad p^{Q}_{1MM}(\hat{s}) \;=\; \frac{1}{2}\hat{s}\; e^{-\frac{1}{4}\hat{s}^2}\;. \tag{4.37}$$

The massive two-flavour density is given by

$$\rho^{(2)}_{1MM}(\hat{x}) = \rho^{Q}_{1MM}(\hat{x}) - \hat{x}\;\frac{\int_0^1 dttJ_0(t\hat{x})I_0(t\hat{m}_1)\int_0^1 dttJ_0(t\hat{x})I_0(t\hat{m}_2)}{\int_0^1 dttI_0(t\hat{m}_1)I_0(t\hat{m}_2)} \tag{4.38}$$

as well as its first eigenvalue distribution by

$$p^{(2)}_{1MM}(\hat{s}) = \frac{1}{2}\hat{s}\; e^{-\frac{1}{4}\hat{s}^2}\;\frac{I_2(\hat{m}'_1)\hat{m}'_2 I_3(\hat{m}'_2) - I_2(\hat{m}'_2)\hat{m}'_1 I_3(\hat{m}'_1)}{I_0(\hat{m}_1)\hat{m}_2 I_1(\hat{m}_2) - I_0(\hat{m}_2)\hat{m}_1 I_1(\hat{m}_1)}\;. \tag{4.39}$$

Here primed masses are shifted according to $\hat{m}_i^{'2} \equiv \hat{m}_i^2 + s^2$. We also need the one-flavour density and its first eigenvalue

$$\rho^{(1)}_{1MM}(\hat{x}) = \rho^{Q}_{1MM}(\hat{x}) \;-\; \hat{x}J_0(\hat{x})\;\frac{\int_0^1 dttJ_0(t\hat{x})I_0(t\hat{m}_1)}{I_0(\hat{m}_1)}\,, \tag{4.40}$$

$$p^{(1)}_{1MM}(\hat{s}) = \frac{1}{2}\hat{s}\; e^{-\frac{1}{4}\hat{s}^2}\frac{I_2(\hat{m}_1^{'2})}{I_0(\hat{m}_1)}\;. \tag{4.41}$$

### 4.4 Partial quenching

As another example we can consider the partially quenched gap probability eq. (4.26) with $N_1 = 0$ and one single ($N_2 = 1$) flavour of mass $m_1$,

$$E^{(0+1)}_{S\;0,0}(\hat{s},0) \;\equiv\; \exp\left[-\frac{1}{4}\hat{s}^2 - \frac{1}{2}\hat{\delta}^2\right]\frac{Q_S(\hat{m}_1;t=1)}{I_0(\hat{m}_1)}\,, \tag{4.42}$$

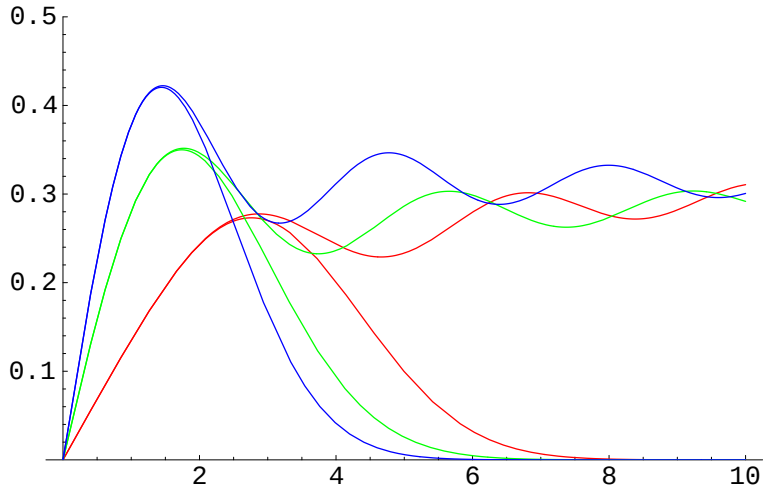


**Figure 4:** The eigenvalue density and first eigenvalue for $N_f = 0 + 2$ with imaginary chemical potential $\hat{\delta} = 1$ (low red), 3 (middle green), and 10 (upper blue curve), at fixed quark masses of flavour $N_2$ $\hat{m}_1 = 3$, $\hat{m}_2 = 4$.

where the extra $\hat{\delta}$-dependent factor $e^{-\hat{\delta}^2/2}$ comes from the partition function $\mathcal{Z}_0^{(0+1)}$ that normalises this gap probability. Its derivative is shown in figure 3 together with the corresponding density [12]

$$\rho_{1,0}^{(0+1)}(\hat{x}) \;=\; \rho_{1MM}^{Q}(\hat{x}) \;-\; \exp\left[-\frac{1}{2}\hat{\delta}^2\right]\hat{x}\frac{J_0(\hat{x})}{I_0(\hat{m}_1)}\int_0^1 dTTe^{\frac{1}{2}T^2\hat{\delta}^2}I_0(T\hat{m}_1)J_0(T\hat{x})\;. \quad (4.43)$$

Again we recover the one-matrix density eq. (4.40) when setting $\hat{\delta} = 0$ in eq. (4.43). For $\hat{\delta} = 1$ the curve is still close to this one-flavour one-matrix result, see figure 2, and for $\hat{\delta} \gg 1$ the curves approach the quenched one-matrix density. This can again be checked analytically by taking $\hat{\delta} \to \infty$ of eq. (4.43) and using eq. (4.36). The exponentials cancel but the prefactor $1/\hat{\delta}^2$ makes the second term in eq. (4.43) vanish, leading to the quenched result. Again these limits $\hat{\delta} \to 0$ and $\hat{\delta} \to \infty$ can also be checked for the gap probability.

As the last and probably most physically relevant example we consider the partially quenched case of $N_1 = 0$ and $N_2 = 2$ flavours with two possibly non-degenerate masses $\hat{m}_1$ and $\hat{m}_2$

$$E_{S\;0,0}^{(0+2)}(\hat{s},0) = \exp\left[-\frac{1}{4}\hat{s}^2 - \hat{\delta}^2\right]\frac{\det\begin{bmatrix} Q_S(\hat{m}_1;t=1) & \partial_t Q_S(\hat{m}_1;t)\big|_{t=1} \\ Q_S(\hat{m}_2;t=1) & \partial_t Q_S(\hat{m}_2;t)\big|_{t=1}\end{bmatrix}}{I_0(\hat{m}_1)\hat{m}_2 I_1(\hat{m}_2) - I_0(\hat{m}_2)\hat{m}_1 I_1(\hat{m}_1)}\;. \quad (4.44)$$

When setting $\hat{\mu}_2 = 0$ to have the sea quarks of flavour $N_2$ free of chemical potential we simply have $\hat{\delta} = -\hat{\mu}_1 \neq 0$. The limit of equal masses can also be taken at the expense of a further derivative within each determinant. As stressed before, this case should be particularly useful for lattice gauge theory simulations, since it corresponds to ordinary configurations without chemical potential.



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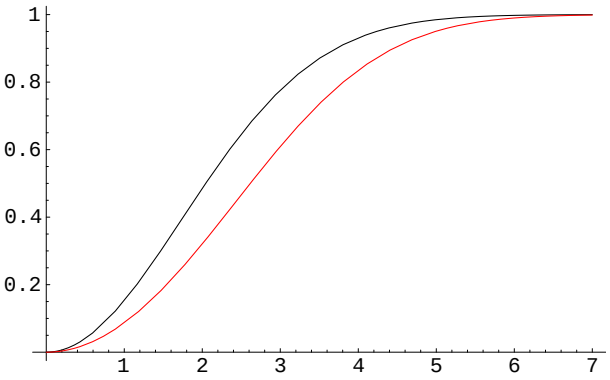


**Figure 5:** The integrated lowest eigenvalue distribution $1 - E_{0,0}(s)$ for $1+1$ flavours eq. (4.33) (lower red) and $0+2$ flavours eq. (4.44) (upper black curve), both at masses $\hat{m}_1 = 3$, $\hat{m}_2 = 4$ and $\hat{\delta} = 3$.

The comparison to the spectral density given by[6]

$$\rho_{1,0}^{(0+2)}(\hat{x}) = \rho_{1MM}^{Q}(\hat{x}) - \exp\left[-\frac{1}{2}\hat{\delta}^2\right]\hat{x}\left(\hat{m}_1 I_1(\hat{m}_1)I_0(\hat{m}_2) - \hat{m}_2 I_0(\hat{m}_1)I_1(\hat{m}_2)\right)^{-1} \tag{4.45}$$
$$\times\left[\int_0^1 dt t e^{\frac{1}{2}\hat{\delta}^2 t^2} J_0(\hat{x}t)I_0(\hat{m}_1 t)\Big(-I_0(\hat{m}_2)(\hat{x}J_1(\hat{x})+\hat{\delta}^2 J_0(\hat{x}))-\hat{m}_2 I_1(\hat{m}_2)J_0(\hat{x})\Big)\right.$$
$$\left.+\int_0^1 dt t e^{\frac{1}{2}\hat{\delta}^2 t^2} J_0(\hat{x}t)I_0(\hat{m}_2 t)\Big(I_0(\hat{m}_1)(\hat{x}J_1(\hat{x})+\hat{\delta}^2 J_0(\hat{x}))+\hat{m}_1 I_1(\hat{m}_1)J_0(\hat{x})\Big)\right],$$

is shown in figure 4. The one-matrix model result with two flavours is again recovered by setting $\hat{\delta} = 0$ given by eq. (4.38). For $\hat{\delta} = 1$ the curve is close to the two-flavour one-matrix result, see figure 2, and for $\hat{\delta} \gg 1$ the curves approach the quenched one-matrix quantities, see figure 2. This matching can once more be checked analytically by taking $\hat{\delta} \to \infty$ of eq. (4.45). Using eq. (4.36) as well as the cancellation of the two terms proportional to $\hat{\delta}^2$ in the last two lines of eq. (4.45) leads again to a complete quenching of all flavours. The same can be checked for the gap probability.

Finally we can also compare directly the gap probabilities in our two-matrix theory for $1+1$ flavours and $0+2$ partially quenched flavours, where in figure 5 we show $1-E_{0,0}(s)$. For $\hat{\delta} = 1$ the difference is still small but it grows rapidly with increasing $\hat{\delta}$ since both theories converge towards different limits for $\hat{\delta} \to \infty$ as was pointed out earlier. Equivalently this results into the following comparison for the densities and first eigenvalue shown in figure 6. It should be noted here that the quantity $1 - E_{0,0}(s)$ is the integrated lowest eigenvalue distribution, by some considered a convenient quantity for comparison with the lattice gauge theory data.

We end this section by pointing out that there is no analogous computation of a $\mu$-dependent Dirac eigenvalue distribution in the fully quenched case. The mixed two-point spectral correlation function has non-trivial $\mu$-dependence [8] (and this dependence allows for the determination of a quenched value of $F_\pi$ using this technique). But the one-point

[6]There is a term missing in the bottom right of the $3\times 3$ matrix in the relevant formula eq. (3.53) of [12].



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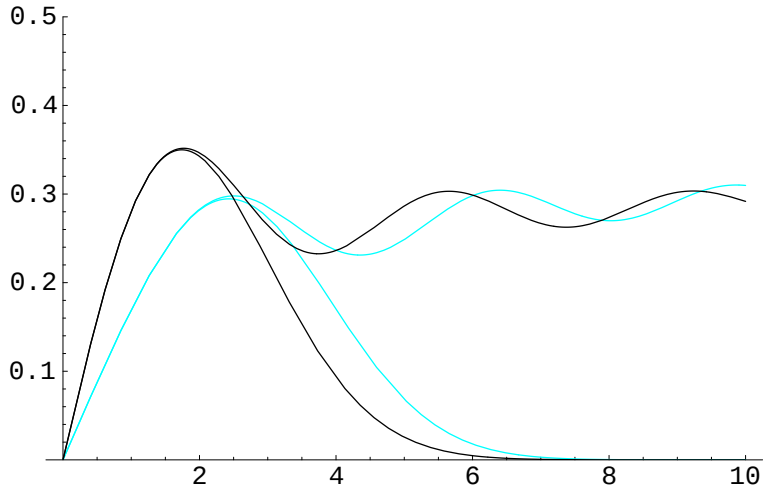


**Figure 6:** The density and first eigenvalue for $1+1$ flavours (right blue) vs. $0+2$ flavours (left black curve), both at masses $\hat{m}_1 = 3$, $\hat{m}_2 = 4$ and fixed $\hat{\delta} = 3$.

function is $\mu$-independent in chiral perturbation theory because it is generated by the addition of just one valence quark; it is $\mu$-independent to all orders in chiral perturbation theory because the valence pions do not carry net baryon charge.

## 5. Conclusions and outlook

The two main results of this paper are the following. We have shown how individual distributions of the lowest-lying eigenvalues of Dirac operators that are subjected to two different external Abelian vector potentials ("imaginary chemical potential") can be derived from field theory. The results have been given in terms of generalised gap probabilities from which the distributions can all be derived.

To compute the gap probabilities from field theory one needs to know spectral correlation functions, all of which can be given a well-defined meaning in the field theoretical setting. In particular, in the scaling region known as the $\epsilon$-regime, these eigenvalue distributions can be derived from the corresponding effective theory, the chiral Lagrangian. To make these computations concrete, we have used the equivalent Random Two-Matrix Theory to derive the distribution of the lowest Dirac operator eigenvalue in the $\epsilon$-regime of QCD with imaginary chemical potential. As stressed in the introduction, these analytical formulas may provide a very convenient way of determining simultaneously the infinite-volume chiral condensate $\Sigma$ and the pion decay constant $F_\pi$ by means of numerical simulations in lattice gauge theory.

We have given explicit formulas for the lowest individual distribution in terms of a new kernel, both in the case of full QCD with imaginary chemical potential, and for the analogue of partially quenched QCD in which quarks are dynamical, but do not carry chemical potential. Especially the latter may provide the most useful formulation in terms of comparisons with numerical lattice data.



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## A. A determinant identity

In this appendix we prove the following identity for any number of $N_2$ flavours

$$\begin{vmatrix} \hat{L}_0^\nu(m_1) & \dots & (1-\tau)^{-k}\hat{L}_k^\nu(m_1) & \dots & (1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(m_1) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(m_{N_2}) & \dots & (1-\tau)^{-k}\hat{L}_k^\nu(m_{N_2}) & \dots & (1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(m_{N_2}) \\ \hat{L}_0^\nu(x_1) & \dots & \hat{L}_k^\nu(x_1) & \dots & \hat{L}_{N+N_2-1}^\nu(x_1) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(x_N) & \dots & \hat{L}_k^\nu(x_N) & \dots & \hat{L}_{N+N_2-1}^\nu(x_N) \end{vmatrix} = \tag{A.1}$$

$$= \begin{vmatrix} \hat{L}_0^\nu(\frac{m_1}{\tau}) & \dots & \tau^k(1-\tau)^{-k}\hat{L}_k^\nu(\frac{m_1}{\tau}) & \dots & \tau^{N+N_2-1}(1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(\frac{m_1}{\tau}) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(\frac{m_{N_2}}{\tau}) & \dots & \tau^k(1-\tau)^{-k}\hat{L}_k^\nu(\frac{m_{N_2}}{\tau}) & \dots & \tau^{N+N_2-1}(1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(\frac{m_{N_2}}{\tau}) \\ 1 & \dots & x_1^k & \dots & x_1^{N+N_2-1} \\ \dots & & \dots & & \dots \\ 1 & \dots & x_N^k & \dots & x_N^{N+N_2-1} \end{vmatrix} .$$

Here we use the notation $\hat{L}_n^\nu(x) = x^n + \cdots$ for the Laguerre polynomials in monic normalisation

$$\hat{L}_n^\nu(x) \equiv (-1)^n n!\; L_n^\nu(x) \;=\; \sum_{j=0}^{n}(-1)^{n+j}\frac{n!(n+\nu)!}{(n-j)!(\nu+j)!j!}\; x^j \quad . \tag{A.2}$$

For simplicity we will prove the identity for one flavour $N_2 = 1$ first, by induction in $N$. For $N = 1$ we have that

$$\begin{vmatrix} \hat{L}_0^\nu(m) & \frac{1}{1-\tau}\hat{L}_1^\nu(m) \\ \hat{L}_0^\nu(x) & \hat{L}_1^\nu(x) \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{1-\tau}(m-\nu-1) \\ 1 & x-\nu-1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{\tau}{1-\tau}(\frac{m}{\tau}-\nu-1) \\ 1 & x \end{vmatrix} = \begin{vmatrix} \hat{L}_0^\nu(\frac{m}{\tau}) & \frac{1}{1-\tau}\hat{L}_1^\nu(\frac{m}{\tau}) \\ 1 & x \end{vmatrix} , \tag{A.3}$$

by adding $\nu+1$ times the first column to the second column. Next we do the induction step,

$$\begin{vmatrix} \hat{L}_0^\nu(m) & \dots & (1-\tau)^{-k}\hat{L}_k^\nu(m) & \dots & (1-\tau)^{-(N+1)}\hat{L}_{N+1}^\nu(m) \\ \hat{L}_0^\nu(x_1) & \dots & \hat{L}_k^\nu(x_1) & \dots & \hat{L}_{N+1}^\nu(x_1) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(x_N) & \dots & \hat{L}_k^\nu(x_N) & \dots & \hat{L}_{N+1}^\nu(x_N) \end{vmatrix} = \tag{A.4}$$

$$= \begin{vmatrix} \hat{L}_0^\nu(\frac{m}{\tau}) & \dots & \tau^k(1-\tau)^{-k}\hat{L}_k^\nu(\frac{m}{\tau}) & \dots & \tau^N(1-\tau)^{-N}\hat{L}_N^\nu(\frac{m}{\tau}) & (1-\tau)^{-(N+1)}\hat{L}_{N+1}^\nu(m) \\ 1 & \dots & x_1^k & \dots & x_1^N & \hat{L}_{N+1}^\nu(x_1) \\ \dots & & \dots & & \dots & \dots \\ 1 & \dots & x_N^k & \dots & x_N^N & \hat{L}_{N+1}^\nu(x_N) \end{vmatrix} .$$

Here we have expanded with respect to the last column and used the induction assumption for $N$, as well as the fact that the sub-determinant containing only $x$-variables of monic Laguerre polynomials can be replaced by the Vandermonde determinant.

To get monic powers in the last column (except in the first element) we subsequently subtract multiples of columns from the left, using eq. (A.2), and we obtain

$$\begin{vmatrix} \hat{L}_0^\nu(\frac{m}{\tau}) & \dots & \tau^k(1-\tau)^{-k}\hat{L}_k^\nu(\frac{m}{\tau}) & \dots & \tau^N(1-\tau)^{-N}\hat{L}_N^\nu(\frac{m}{\tau}) & P(m) \\ 1 & \dots & x_1^k & \dots & x_1^N & x_1^{N+1} \\ \dots & & \dots & & \dots & \dots \\ 1 & \dots & x_N^k & \dots & x_N^N & x_N^{N+1} \end{vmatrix} . \tag{A.5}$$

The first element in the last column now reads

$$P(m) \;=\; (1-\tau)^{-(N+1)}\hat{L}_{N+1}^\nu(m) - \sum_{j=0}^{N}(-1)^{N+1+j}\frac{(N+1)!(N+1+\nu)!}{(N+1-j)!(\nu+j)!j!}\frac{\tau^j}{(1-\tau)^j}\hat{L}_j^\nu\left(\frac{m}{\tau}\right) . \tag{A.6}$$

As a last step we need to show that $P(m) = \tau^{N+1}(1-\tau)^{-(N+1)}\hat{L}_{N+1}^\nu(\frac{m}{\tau})$. This relation holds due to the following identity [23], which can be easily proven by induction. It is expressed in terms of usual non-monic Laguerre polynomials

$$L_{N+1}^\nu(m) = \sum_{j=0}^{N+1}\frac{(N+1+\nu)!}{(N+1-j)!(\nu+j)!}\tau^j(1-\tau)^{N+1-j}L_j^\nu\left(\frac{m}{\tau}\right) , \tag{A.7}$$

which finishes the first part of our proof. As a remark which is useful for the main text this identity is usually quoted as [23]

$$L_n^\nu(zw) = \sum_{j=0}^{n}\frac{(n+\nu)!}{(n-j)!(\nu+j)!}w^j(1-w)^{n-j}L_j^\nu(z) . \tag{A.8}$$

In the above it was not essential in the manipulation of columns that we had one mass flavour $N_2 = 1$ only. We can in fact do an inductive proof in the column number $k$ for any $N_2$ and $N$,

$$\begin{vmatrix} \hat{L}_0^\nu(m_1) & \dots & (1-\tau)^{-k}\hat{L}_k^\nu(m_1) & \dots & (1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(m_1) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(m_{N_2}) & \dots & (1-\tau)^{-k}\hat{L}_k^\nu(m_{N_2}) & \dots & (1-\tau)^{-(N+N_2-1)}\hat{L}_{N+N_2-1}^\nu(m_{N_2}) \\ \hat{L}_0^\nu(x_1) & \dots & \hat{L}_k^\nu(x_1) & \dots & \hat{L}_{N+N_2-1}^\nu(x_1) \\ \dots & & \dots & & \dots \\ \hat{L}_0^\nu(x_N) & \dots & \hat{L}_k^\nu(x_N) & \dots & \hat{L}_{N+N_2-1}^\nu(x_N) \end{vmatrix} = \tag{A.9}$$

$$= \begin{vmatrix} \hat{L}_0^\nu(\frac{m_1}{\tau}) & \dots & \frac{\tau^k}{(1-\tau)^k}\hat{L}_k^\nu(\frac{m_1}{\tau}) & \frac{1}{(1-\tau)^{k+1}}\hat{L}_{k+1}^\nu(m_1) & \dots & \frac{1}{(1-\tau)^{N+N_2-1}}\hat{L}_{N+N_2-1}^\nu(m_1) \\ \dots & & \dots & \dots & & \dots \\ \hat{L}_0^\nu(\frac{m_{N_2}}{\tau}) & \dots & \frac{\tau^k}{(1-\tau)^k}\hat{L}_k^\nu(\frac{m_{N_2}}{\tau}) & \frac{1}{(1-\tau)^{k+1}}\hat{L}_{k+1}^\nu(m_{N_2}) & \dots & \frac{1}{(1-\tau)^{N+N_2-1}}\hat{L}_{N+N_2-1}^\nu(m_{N_2}) \\ 1 & \dots & x_1^k & \hat{L}_{k+1}^\nu(x_1) & \dots & \hat{L}_{N+N_2-1}^\nu(x_1) \\ \dots & & \dots & \dots & & \dots \\ 1 & \dots & x_N^k & \hat{L}_{k+1}^\nu(x_N) & \dots & \hat{L}_{N+N_2-1}^\nu(x_N) \end{vmatrix} .$$



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The start for $k = 1$ is trivially true in analogy to eq. (A.3). The induction step from $k$ to $k+1$ easily follows by subtracting the left columns for $l \leq k$ from column $k+1$, and using again eq. (A.7) for $N+1 \to k+1$. Putting $k = N + N_2$ ends the proof.

## B. An identity for Laguerre polynomials

The relation we show here is given in eq. (4.14),

$$L_m^{-1}(x) \;=\; \sum_{j=0}^{m} (-)^j \binom{j+k}{k} L_{m-j}^{j+k}(x)\,, \tag{B.1}$$

where the right hand side is independent of $k$. It follows from a known identity eq. (4.4.1.14) in [21]

$$\frac{1}{(\beta)_m} L_m^{\alpha+\beta-1}(x) \;=\; \sum_{i=0}^{m} \frac{1}{(m-i)!(\beta)_i} L_i^{\alpha-i}(x)\;. \tag{B.2}$$

Here $(\beta)_m$ is the Pochhammer symbol. In choosing $\alpha = -\beta$ we obtain

$$\begin{aligned} \frac{(-)^m(\alpha-m)!}{\alpha!} L_m^{-1}(x) &= \sum_{i=0}^{m} \frac{(-)^i(\alpha-i)!}{(m-i)!\alpha!} L_i^{\alpha-i}(x)\;. \\ &= \sum_{j=0}^{m} \frac{(-)^{m-j}(\alpha-m+j)!}{j!\alpha!} L_{m-j}^{\alpha-m+j}(x)\;. \end{aligned} \tag{B.3}$$

In the second step we have changed summation from $i$ to $j = m - i$. Finally choosing $\alpha = m + k$ we obtain eq. (B.1) above.

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